101.29 Sums of square roots

Introduction

Sums of consecutive integral roots have been studied by many mathematicians, for instance, the following identities are known.

(1) $\lfloor\sqrt{n} + \sqrt{n+1}\rfloor = \lfloor\sqrt{4n+1}\rfloor$

(2) $\lfloor\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\rfloor = \lfloor\sqrt{9n+8}\rfloor$

(3) $\lfloor\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3}\rfloor = \lfloor\sqrt{16n+20}\rfloor$

(4) $\lfloor\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4}\rfloor = \lfloor\sqrt{25n+49}\rfloor$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

The formula (1) is folklore; (2) was posed by F. D. Hammer as Problem E3010 in *The American Mathematical Monthly*, see three different methods in [1]; (3) was published by Z. Wang in [2]; and (4) was proved by X. Zhan in [3].

In 2008, P. W. Saltzman and P. Yuan, see [4, Lemma 2.2], showed that if $n > \dfrac{k^2(k-1)(2k-1)}{24}$ for any integer $k \geqslant 2$, then

$$\lfloor\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \ \dots\ + \sqrt{n+k-1}\rfloor = \left\lfloor\sqrt{k^2 n + \frac{(k-1)k^2}{2} - 1}\right\rfloor. \quad (1)$$

In this Note, we present a simple alternative proof of (1) and extend this result to sums of square roots of sequences of non-negative real numbers.

Main Results

Throughout this section, let $k \geqslant 2$ be a fixed positive integer. We first derive a lemma which is the core of our main result.

Lemma 1: Let $a_1, a_2, \dots, a_k$ be non-negative real numbers which are not all equal. Let $A = \frac{1}{2}\min\limits_{i \neq j}\{a_i + a_j\}$. Then the following statements hold:

(i) $\sqrt{n+a_1} + \sqrt{n+a_2} + \ \dots\ + \sqrt{n+a_k} < \sqrt{k^2 n + k\sum\limits_{i=1}^{k} a_i}$ for all positive integers n,

(ii) $\sqrt{n+a_1} + \sqrt{n+a_2} + \ \dots\ + \sqrt{n+a_k} > \sqrt{k^2 n + k\sum\limits_{i=1}^{k} a_i - 1}$ for all positive integers $n \geqslant \dfrac{1}{4}\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}(a_i - a_j)^2 - A$.

Proof: Observe that

$$\left(\sum_{i=1}^{k}\sqrt{n+a_i}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\sqrt{(n+a_i)(n+a_j)}.$$

https://doi.org/10.1017/mag.2017.131 Published online by Cambridge University Press

(i) By the Arithmetic Mean−Geometric Mean inequality [5, Chapter 2]

$$\sqrt{xy} \leqslant \frac{x + y}{2}$$

(where the equality holds if, and only if, $x = y$), with $x = n + a_i$ and $y = n + a_j$, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{(n + a_i)(n + a_j)} < \sum_{i=1}^{k} \sum_{j=1}^{k} \left(n + \frac{a_i + a_j}{2}\right)$$

$$= k^2 n + k \sum_{i=1}^{k} a_i.$$

Thus,

$$\sum_{i=1}^{k} \sqrt{n + a_i} < \sqrt{k^2 n + k \sum_{i=1}^{k} a_i}.$$

(ii) By the Geometric Mean-Harmonic Mean inequality [5, Chapter 2]

$$\sqrt{xy} \geqslant \frac{xy}{\frac{1}{2}(x + y)}$$

(where the equality holds if, and only if, $x = y$), with $x = n + a_i$ and $y = n + a_j$, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{(n + a_i)(n + a_j)} > \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(n + a_i)(n + a_j)}{n + \frac{1}{2}(a_i + a_j)}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\left(n + \frac{1}{2}(a_i + a_j)\right)^2 - \left(\frac{1}{2}(a_i + a_j)\right)^2 + a_i a_j}{n + \frac{1}{2}(a_i + a_j)}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \left(n + \frac{a_i + a_j}{2} - \frac{(a_i - a_j)^2}{4n + 2(a_i + a_j)}\right)$$

$$= k^2 n + k \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(a_i - a_j)^2}{4n + 2(a_i + a_j)}.$$

Thus,

$$\sum_{i=1}^{k} \sqrt{n + a_i} > \sqrt{k^2 n + k \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(a_i - a_j)^2}{4n + 2(a_i + a_j)}}.$$

Now, assume that $n \geqslant \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} (a_i - a_j)^2 - A$. We get

$$1 \geqslant \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(a_i - a_j)^2}{4n + 4A} > \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(a_i - a_j)^2}{4n + 2(a_i + a_j)}.$$

https://doi.org/10.1017/mag.2017.131 Published online by Cambridge University Press

Therefore,

$$\sum_{i=1}^{k} \sqrt{n + a_i} > \sqrt{k^2 n + k \sum_{i=1}^{k} a_i - 1}$$

for $n \geqslant \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} (a_i - a_j)^2 - A$.

As a consequence of Lemma 1, we obtain the following result.

Theorem 1: Let $a_1, a_2, \ldots, a_k$ be non-negative real numbers which are not all equal and $k \sum_{i=1}^{k} a_i$ a positive integer. Let $A = \frac{1}{2} \min_{i \neq j} \{a_i + a_j\}$. Then for all positive integers $n \geqslant \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} (a_i - a_j)^2 - A$,

$$\left\lfloor \sqrt{n + a_1} + \sqrt{n + a_2} + \ldots + \sqrt{n + a_k} \right\rfloor = \left\lfloor \sqrt{k^2 n + k \sum_{i=1}^{k} a_i - 1} \right\rfloor. \qquad (2)$$

Proof: Let $P = k^2 n + k \sum_{i=1}^{k} a_i$ and $n \geqslant \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} (a_i - a_j)^2 - A$. Suppose that equation (2) does not hold. By Lemma 1(ii), there is a positive integer t such that

$$\sqrt{P - 1} < t \leqslant \sqrt{n + a_1} + \sqrt{n + a_2} + \ldots + \sqrt{n + a_k}.$$

By Lemma 1(i), we obtain

$$\sqrt{P - 1} < t < \sqrt{P}.$$

Equivalently,

$$P - 1 < t^2 < P,$$

contradicting the fact that there is no integer between two consecutive integers.

Corollary 1: Let m be a rational number with mk an integer. Then for all positive integers $n \geqslant \dfrac{m^2 k^2 (k^2 - 1) - 12m}{24}$,

$$\left\lfloor \sqrt{n} + \sqrt{n + m} + \sqrt{n + 2m} + \ldots + \sqrt{n + (k - 1)m} \right\rfloor = \left\lfloor \sqrt{k^2 n + \frac{mk^2 (k^2 - 1)}{2} - 1} \right\rfloor.$$

Proof: Put $a_i = m(i - 1)$ in Theorem 1, we get that $k \sum_{i=1}^{k} a_i = \dfrac{mk^2 (k - 1)}{2}$ is a positive integer and $A = \frac{1}{2} m$. Moreover,

https://doi.org/10.1017/mag.2017.131 Published online by Cambridge University Press

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(a_i - a_j)^2 = m^2\sum_{i=0}^{k-1}\sum_{j=0}^{m-1}(i - j)^2$$
$$= 2m^2k\sum_{i=0}^{k-1}i^2 - 2m^2\left(\sum_{i=0}^{m-1}i\right)$$
$$= \frac{m^2k^2(k-1)(2k-1)}{3} - \frac{m^2k^2(k-1)^2}{2}$$
$$= \frac{m^2k^2(k-1)^2}{6}.$$

Hence, we obtain the desired result.

Note: Substituting $m = 1$ in Corollary 1, we obtain that identity (1) holds for $n \geqslant \frac{k^2(k^2 - 1) - 12}{24}$. One can see that our lower bound on n is approximately half of the one obtained in [4].

Putting $m = 2, \frac{1}{k}$ in Corollary 1, we respectively get parts (i), (ii) as shown in the following example.

Example 1: Let n be a positive integer.

(i) If $n \geqslant \frac{k^2(k^2 - 1)}{6} - 1$, then

$$\left\lfloor\sqrt{n} + \sqrt{n+2} + \sqrt{n+4} + \ \dots \ + \sqrt{n+2(k-1)}\right\rfloor = \left\lfloor\sqrt{k^2(n+k-1)-1}\right\rfloor.$$

(ii) If $n \geqslant \frac{k^3 - k - 12}{24k}$, then

$$\left\lfloor\sqrt{n} + \sqrt{n+\frac{1}{k}} + \sqrt{n+\frac{2}{k}} + \ \dots \ + \sqrt{n+\frac{k-1}{k}}\right\rfloor = \left\lfloor\sqrt{k^2n + \frac{k(k-1)}{2} - 1}\right\rfloor.$$

Let $\{F_n\}_{n\geqslant 0}$ and $\{L_n\}_{n\geqslant 0}$ be the sequences of the Fibonacci and Lucas numbers, respectively, which are given by

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1},$$
$$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}.$$

In Theorem 1, let $k = 2$ and a_i be the i th Fibonacci number and the i th Lucas number, respectively. We get the following example.

Example 2: Let n and i be positive integers. Then

$$\left\lfloor\sqrt{n+F_i} + \sqrt{n+F_{i+1}}\right\rfloor = \left\lfloor\sqrt{4n+2F_{i+2}-1}\right\rfloor \qquad \text{if } n \geqslant \tfrac{1}{2}\left(F_{i-1}^2 - F_{i+2}\right),$$

and

$$\left\lfloor\sqrt{n+L_i} + \sqrt{n+L_{i+1}}\right\rfloor = \left\lfloor\sqrt{4n+2L_{i+2}-1}\right\rfloor \qquad \text{if } n \geqslant \tfrac{1}{2}\left(L_{i-1}^2 - L_{i+2}\right).$$

Putting $a_1 = 0$ and $a_i = \frac{1}{i(i-1)}$ for $2 \leqslant i \leqslant k$ in Theorem 1 and manually checking the first few values of n, we get the following example.

Example 3: Let n and k be positive integers. Then for $2 \leqslant k \leqslant 10$

$$\left\lfloor \sqrt{n} + \sqrt{n + \frac{1}{1 \times 2}} + \sqrt{n + \frac{1}{2 \times 3}} + \dots + \sqrt{n + \frac{1}{(k-1)k}} \right\rfloor = \left\lfloor \sqrt{k^2 n + k - 2} \right\rfloor.$$

Acknowledgments

The authors are grateful to the anonymous referee for the careful reading and helpful comments. This research was supported by the Kasetsart University Research and Development Institute (KURDI), Thailand.

References

1. F. D. Hammer, Problem E3010, *Amer. Math. Monthly* **95**(2) (1988), pp. 133-134.
2. Z. Wang, *A City of Nice Mathematics* (in Chinese), The Democracy and Construction Press, Beijing (2000).
3. X. Zhan, Formulae for sums of consecutive square roots, *Math. Intelligencer* **27** (2005) pp. 4-5.
4. P. W. Saltzman and P. Yuan, On sums of consecutive integral roots, *Amer Math. Monthly* **115**(3) (2008) pp. 254-261.
5. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press (1952).

10.1017/mag.2017.131

KANTAPHON KUHAPATANAKUL and PONGPOL RUANKONG

Department of Mathematics, Kasetsart University, Bangkok 10900, Thailand

e-mail: *fscikpkk@ku.ac.th*; *fscippru@ku.ac.th*

101.30 A few remarks concerning a class of infinite sums

This Note derives a general expression for the infinite sums of fractions having numerator 1 and denominators that are the product of a sequence of numbers. The sequences are all arithmetic progressions having a common difference d, with a starting value q and the number of terms p. By entering values for d, q and p and forming the definite integral for the interval 0 to 1 of the Abel power series in x the sum is returned.

Part 1: In which sequences of numbers with unit differences starting with 1 are used as denominators.

Consider the infinite summation of a series of terms such as the example $\frac{1}{1 \times 2 \times 3 \times 4} + \frac{1}{5 \times 6 \times 7 \times 8} + \dots$, where the denominators form an unbroken sequence.